Theorem: 8

Second mean value theorem for R.S.I

Assume that α is continuous and that f increasing on $[a,b]$ then there exists a points x_0 in $[a,b]$ such that $\int_a^b f(x)\,d\alpha(x) = f(a)\int_a^{x_0} d\alpha(x) + f(b)\int_{x_0}^b d\alpha(x)$

proof:

Given:

(i) α is continuous

(ii) f increasing on $[a,b]$

Since f increasing on $[a,b]$, f is bounded variation on $[a,b]$ [By thm 7 of unit 1]

Since α is continuous on $[a,b]$ and f is of bounded variation on $[a,b]$, $f \in R(\alpha)$ on $[a,b]$ [By note under thm 4 of unit 2]

$\therefore$ By integrating by parts, we have

$\alpha \in R(f)$ on $[a,b]$

and $\int_a^b f\,d\alpha + \int_a^b \alpha\,df = f(b)\alpha(b) - f(a)\alpha(a) \longrightarrow (1)$

f is increasing on $[a,b]$ and $\alpha \in R(f)$ on $[a,b]$ and α is continuous on $[a,b]$.

$\Rightarrow$ By First mean value theorem for R.S.I

$\exists$ a point $x_0 \in [a,b]$ such that

$$\int_a^b \alpha\,df = \alpha(x_0)\,[f(b) - f(a)] \longrightarrow (2)$$

Using (2) in (1) we have,

$$\int_a^b f\,d\alpha = f(b)\alpha(b) - f(a)\alpha(a) - \alpha(x_0)[f(b) - f(a)]$$

$$= f(b)\alpha(b) - f(a)\alpha(a) - f(b)\alpha(x_0) + f(a)\alpha(x_0)$$

$$= f(a)[\alpha(x_0) - \alpha(a)] + f(b)[\alpha(b) - \alpha(x_0)]$$

$$\int_a^b f\,d\alpha = f(a)\int_a^{x_0} d\alpha(x) + f(b)\int_{x_0}^b d\alpha(x) \quad \text{for some } x_0 \in [a,b]$$

Hence the proof.

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Theorem: 9 First Fundamental Integral theorem of calculus.

The integral as a function of the interval

Let α be of bounded variation $\alpha(b)-\alpha(a)\geq 0$ on $[a,b]$ and assume that let $f \in R(\alpha)$ on $[a,b]$

Define F by the equation,

$$F(x) = \int_a^x f(x)\, d\alpha(x) \text{ if } x \in [a,b]$$

Then we have,

i) F is of bounded variation on $[a,b]$

ii) Every point of continuity of α is also a point of continuity of F.

(iii) If α increasing on $[a,b]$, the derivative $F(x)$ exists at each point ~~of~~ x in (a,b) where $\alpha'(x)$ exists and where f is continuous.

For such x, we have

$$F'(x) = f(x)\, \alpha'(x)$$

Proof: Given,

i) α is of bounded variation on $[a,b]$

ii) $f \in R(\alpha)$ on $[a,b]$

iii) Define $F(x) = \int_a^x f(x)\, d\alpha(x)$, if $x \in [a,b]$

It suffices to prove the theorem when α is increasing on $[a,b]$

By the First mean Value theorem of R.S.I

Since α increasing on $[a,b]$ and $f \in R(\alpha)$ on $[a,b]$

and, $m = \inf\{f(x) : x \in [a,b]\}$ and

$M = \sup\{f(x) : x \in [a,b]\}$ there exists a real number C such that $m \leq c \leq M$ ⟶ (1)

$$\int_a^b f(x)\, d\alpha(x) = c\,[\alpha(b) - \alpha(a)] \longrightarrow (2)$$

and if f is continuous on $[a,b]$, $c = f(x)$, for some $x \in [a,b]$.

(i) to prove: F is of b.v on $[a,b]$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$

consider,
$$\Delta F_k = F(x_k) - F(x_{k-1})$$
$$= \int_a^{x_k} f(x)\, d\alpha(x) - \int_a^{x_{k-1}} f(x)\, d\alpha(x)$$
$$= \int_a^{x_k} f(x)\, d\alpha(x) + \int_{x_{k-1}}^{a} f(x)\, d\alpha(x)$$
$$\Delta F_k = \int_{x_{k-1}}^{x_k} f(x)\, d\alpha(x)$$
$$= c\,[\alpha(x_k) - \alpha(x_{k-1})] \quad \text{By } (2)$$
$$= c \cdot \Delta\alpha_k \quad \boxed{c \le M}$$
$$\le M \cdot \Delta\alpha_k \quad \text{By } (1)$$
$$\Rightarrow \Delta F_k \le M \cdot \Delta\alpha_k$$
$$\Rightarrow \sum_{k=1}^{n} |\Delta F_k| \le M \sum_{k=1}^{n} |\Delta\alpha_k|$$
$$= M \sum_{k=1}^{n} \Delta\alpha_k \quad \{\because \alpha \uparrow \text{ on } [a,b]\}$$
$$= M\,[\alpha(b) - \alpha(a)]$$
$$\sum_{k=1}^{n} |\Delta F_k| \le M\,[\alpha(b) - \alpha(a)], \text{ a +ve constant,}$$

$\therefore F$ is of b.v on $[a,b]$

To prove: Every point of continuity of α is also a point of continuity of F

Let α be continuous at x

$$\alpha(x) = \lim_{y \to x} \left[\frac{\alpha(y) - \alpha(x)}{y - x} \right]$$

α is conti $\Rightarrow$ derivative exist

Then $\lim_{y \to x} \alpha(y) = \alpha(x)$

To prove: F is continuous at x

ie) To prove that $\lim_{y \to x} F(y) = F(x)$

consider,

$$F(y) - F(x) = \int_a^y f(x)\, d\alpha(x) - \int_a^x f(x)\, d\alpha(x)$$

$$F = \int_x^y f(x)\, d\alpha(x) \quad \text{using I mean value theorem}$$

$$\Rightarrow F(y) - F(x) = c\,[\alpha(y) - \alpha(x)]$$

$$\Rightarrow \lim_{y \to x} [F(y) - F(x)] = \lim_{y \to x} [c\,(\alpha(y) - \alpha(x))]$$

$$\Rightarrow \lim_{y \to x} F(y) - \lim_{y \to x} F(x) = \lim_{y \to x} c \left[\lim_{y \to x} \alpha(y) - \lim_{y \to x} \alpha(x) \right]$$

$$\Rightarrow \lim_{y \to x} F(y) - F(x) = \lim_{y \to x} c\, \{\alpha(x) - \alpha(x)\}$$

$$= \lim_{y \to x} c \cdot \{0\}$$

$$\Rightarrow \lim_{y \to x} F(y) - F(x) = 0$$

$$\Rightarrow \lim_{y \to x} F(y) = F(x)$$

$\Rightarrow F$ is continuous at x

Since x is arbitrary, every point of continuity of α is also a point of continuity of F.

(iii) To prove that:

$F'(x)$ exists at each point x in (a, b) where $\alpha'(x)$ exists and where f is continuous

Let f be continuous at $x \neq y$ then $c = f(x)$, $x \in [a, b]$

Let $\alpha'(x)$ exists (ie) α has a derivative at 'x'

(ie) $\alpha'(x) = \lim\limits_{y \to x} \dfrac{\alpha(y) - \alpha(x)}{y - x}$ exists $\longrightarrow$ (1)

we have,

$$F'(x) = \lim_{y \to x} \frac{F(y) - F(x)}{y - x} \longrightarrow (2)$$

Consider,

$$F(y) - F(x) = \int_a^y f\,d\alpha - \int_a^x f\,d\alpha$$

$$F(y) - F(x) = \int_x^y f\,d\alpha$$

$$F(y) - F(x) = c\,[\alpha(y) - \alpha(x)]$$

Divide $(y - x)$ on both sides

$$\frac{F(y) - F(x)}{y - x} = \frac{c\,[\alpha(y) - \alpha(x)]}{y - x}$$

$$\lim_{y \to x} \left\{ \frac{F(y) - F(x)}{y - x} \right\} = \lim_{y \to x} \left\{ \frac{c\,[\alpha(y) - \alpha(x)]}{y - x} \right\}$$

$$\Rightarrow F'(x) = \lim_{y \to x} c \cdot \lim_{y \to x} \left\{ \frac{\alpha(y) - \alpha(x)}{y - x} \right\}$$

$$F'(x) = f(x) \cdot \alpha'(x)$$

Since $f(x)$ and $\alpha'(x)$ exists, $F'(x)$ is also exists

Note:

In part (ii) of theorem : 9 Set $\alpha(x) = x$, $\forall x \in [a, b]$ then $\alpha'(x) = 1$, $\forall x$

we have f is continuous and α' exists $F(x)$ is differentiable.

$F'(x) = f(x) \cdot 1 = f(x)$

Therefore $F'(x) = f(x) \cdot 1 = f(x)$

This result is called the first fundamental theorem of integral calculus.

ie) Let f be continuous on $[a,b]$ and let

$F(x) = \int_a^x f(t)\,dt$, $a < x < b$ then F is differentiable and $F'(x) = f(x)$ at each point of continuity.

Theorem: 10

If $f \in R$ and $g \in R$ on $[a,b]$, let $F(x) = \int_a^x f(t)\,dt$

$G(x) = \int_a^x g(t)\,dt$, if $x \in [a,b]$ then F and G are continuous function of bounded Variation on $[a,b]$.

Also $f \in R(G)$ and $g \in R(F)$ and we have,

$$\int_a^b f(x)\, g(x)\, dx = \int_a^b f(x)\, dG(x)$$

$$= \int_a^b g(x)\, dF(x)$$

Proof: Since $\alpha(x) = x$, α is increasing and α is continuous at every x,

Since $f \in R$ and $g \in R$ on $[a,b]$ and

$F(x) = \int_a^x f(t)\,dt$ and $G(x) = \int_a^x g(t)\,dt$

by part (i) and (ii) of theorem 9,

F and G are functions of bounded Variation on $[a,b]$ and F and G are continuous on $[a,b]$

$\}$ $\because \alpha(x) = x$ is cont on $[a,b]$

Then F and G are continuous functions of b.v on $[a,b]$

The existence of the integrals and ... formula for $\int_a^b f(x)\, g(x)\, d\alpha(x)$ follow by taking $\alpha(x) = x$ in thm : 3 of unit 3.

Section 7.2

Theorem : 11

Change of Variable N Riemann Integral :

Assume that g has a continuous derivative g' on an interval $[c,d]$

Let f be continuous on $g([c,d])$ and define F by the equation,

$$F(x) = \int_{g(c)}^{x} f(t)\, dt, \quad \text{if } x \in g([c,d])$$

then for each x in $[c,d]$, the integral,

$\int_c^x f\, g(t)\, g'(t)\, dt$ exists and has the value $F(g(x))$ In particular, we have.

$$\int_{g(c)}^{g(d)} f(x)\, dx = \int_c^d f(g(t))\, g'(t)\, dt.$$

Proof: Given,

i) g has a continuous derivative g' on $[c,d]$

(ii) f is continuous on $g([c,d])$

(iii) $F(x) = \int_{g(c)}^{x} f(t)\, dt, \quad \forall\, x \in g[c,d]$

Since g' exists on $[c,d]$, g is continuous on $[c,d]$.

Since f is continuous on $g([c,d])$

$f \cdot g$ is continuous on $[c,d]$

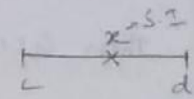
Since g' is continuous on $[c,d]$

$(f\cdot g)g'$ is continuous on $[c,d]$

If $x \in [c,d]$, $(f\cdot g)g'$ is continuous on $[c,x]$

then $\int_c^x (f\cdot g)(t)\, g'(t)\, dt$ exists   [By thm 5 of unit 3]

$\int_c^x f(g(t))\, g'(t)\, dt$ exists



let $G(x) = \int_c^x f(g(t))\, g'(t)\, dt \longrightarrow (*)$

TP: $G(x) = F(g(x)) \longrightarrow ①$

For every $x \in [c,d]$, $G'(x) = F'(g(x))g'(x) \longrightarrow ①$

$\Rightarrow G'(x) = f(g(x))\, g'(x) \longrightarrow ①$

[By 1st fundamental thm of integral calculus]
$\because F'(x) = f(x)$

Also by chain rule,

$(F(g(x)))' = F'(g(x))\, g'(x) \longrightarrow ②$

Given: $F(x) = \int_{g(c)}^{x} f(t)\, dt$

Then by First fundamental theorem of Integral calculus, $F'(x) = f(x)$

$F'(g(x)) = f(g(x)) \longrightarrow ③ \quad \forall x \in [c,d]$

$\{\because g(x) \in g[c,d]\}$

Using ③ in ② (sub)

$(F(g(x)))' = f(g(x))\cdot g'(x) \longrightarrow ④$

① and ④ implies,

$G'(x) - (F(g(x)))'$, $\forall x \in [c,d]$

$\Rightarrow G'(x) - (F(g(x)))' = 0$, $\forall x \in [c,d]$

$\Rightarrow [G - F(g)]'(x) = 0$, $\forall x \in [c,d]$

Integrating,

$\Rightarrow [G - F(g)](x) = k$, a constant, $\forall x \in [c,d] \longrightarrow ⑤$

from (*)

when $x = c$

$G(x) = \int_c^x f(g(t))\, g'(t)\, dt$

$\Rightarrow G(c) = \int_c^c f(g(t))\, g'(t)\, dt = 0$

And $F(g(x)) = \int_{g(c)}^{g(x)} f(t)\,dt$ | $F(g(x)) = \int_c^x f(g(t))\,g'(t)\,dt$

$G(x) = F(g(x))$

$\Rightarrow F(g(c)) = \int_{g(c)}^{g(c)} f(t)\,dt = 0$

Thus $G(c) = 0$ and $F(g(c)) = 0$ if $x = c$

$\Rightarrow G(x) - F(g(x)) = 0 \longrightarrow (6)$

$\Rightarrow G(x) = F(g(x)), \forall x \in [c,d]$

$\Rightarrow F(g(x)) = \int_c^x f(g(t))\,g'(t)\,dt$

when $x = d$,

(6) $\Rightarrow G(d) = F(g(d))$, $g'(d)$

$\Rightarrow \int_c^d f(g(t))\,g'(t)\,dt = \int_{g(c)}^{g(d)} f(t)\,dt$ Since t is a dummy variable of integration.

$$\int_{g(c)}^{g(d)} f(x)\,dx = \int_c^d f(g(x))\,g'(x)\,dx$$

Hence the proof.

Section 7.28

Theorem : 12.

Second Mean Value theorem for Riemann-Integrals: (or) State and prove Bonnet's theorem:

Let g be continuous and assume that f is increasing on $[a,b]$. Let A and B be two real numbers, satisfy the inequalities,

$$A \le f(a+) \text{ and } B \ge f(b-)$$

Then there exist a point x_0 in $[a,b]$, Such that

(i) $\int_a^b f(x)\,g(x)\,dx = A \int_a^{x_0} g(x)\,dx + B \int_{x_0}^b g(x)\,dx$

In particular if $f(x) \ge 0$, $\forall x$ in $[a,b]$

we have $f(a) = 0$, $A = 0$

(ii) $\int_a^b f(x)\, g(x)\, dx = B \int_{x_0}^{b} g(x)\, dx$, where $x_0 \in [a,b]$

Note: part (b) is known as Bonnet's theorem

proof:

Given

(i) g is continuous on $[a,b]$

(ii) f is increasing on $[a,b]$

(iii) A and B are two real numbers

Satisfying $A \le f(a+)$ and $B \ge f(b-)$

To prove (i)

(a) $\exists$ a point x_0 in $[a,b]$ such that

$$\int_a^b f(x)\, g(x)\, dx = A \int_a^{x_0} g(x)\, dx + B \int_{x_0}^{b} g(x)\, dx$$

If $\alpha(x) = \int_a^x g(t)\, dt$

So

$\alpha'(x) = g(x)$, $\forall$ x in $[a,b]$ $\longrightarrow$ (1)

[By 2nd Fundamental thm of Int. calculus]

Since g is continuous, α' is continuous on $[a,b]$.

α is continuous on $[a,b]$

Since α is continuous on $[a,b]$ and f increasing on $[a,b]$, By Second mean value theorem for R-S I

$\exists$ a point $x_0 \in [a,b]$ such that,

$$\int_a^b f(x)\, d\alpha(x) = f(a) \int_a^{x_0} d\,\alpha(x) + f(b) \int_{x_0}^{b} d\,\alpha(x)$$

$$\Rightarrow \int_a^b f(x)\, \alpha'(x)\, dx = f(a) \int_a^{x_0} \alpha'(x)\, dx + f(b) \int_{x_0}^{b} \alpha'(x)\, dx$$

$$\Rightarrow \int_a^b f(x)\, g(x)\, dx = f(a) \int_a^{x_0} g(x)\, dx + f(b) \int_{x_0}^{b} g(x)\, dx$$

By (1) $\longrightarrow$ (2)